Invariants of Differential Geometry by the Use of Vector Forms.

By C. D. Rice.

I. *Introduction.*

Let the equation of the surface be given by

$$x=f(uv), \tag{1}$$

where u and v are scalar variables. Partial derivatives in what follows with respect to u and v are represented by the subscripts 1 and 2 respectively. Derivatives with respect to s, the length of a curve on the surface, will be denoted by primes. The well-known constants in a point are given by

$$\left.\begin{array}{lll} E=-Sx_1x_1, & F=-Sx_1x_2, & G=-Sx_2x_2, \\ L=Sa_1x_1=-Sax_{11}, & M=Sa_1x_2=Sa_2x_1=-Sax_{12}, & N=Sa_2x_2=-Sax_{22}, \end{array}\right\} \tag{2}$$

where a is the unit vector normal to the surface at the point x. The vectors x', x_1, x_2, a', a_1, a_2 are all parallel to the tangent plane at the point. When a curve upon the surface is determined by a scalar relation

$$\phi(uv)=c,$$

we have

$$\phi_1u'+\phi_2v'=0,$$

or

$$\frac{\phi_2}{u'}=-\frac{\phi_1}{v'}=r. \tag{3}$$

By the use of these relations we find from

$$x'=x_1u'+x_2v'$$

the relation

$$rx'=\phi_2x_1-\phi_1x_2=w. \tag{4}$$

In like manner we find

$$ra'=\phi_2a_1-\phi_1a_2=\overline{\omega}. \tag{5}$$

From the study of surfaces we have

$$\Delta^2=EG-F^2=Sx_1x_1Sx_2x_2-Sx_1x_2Sx_1x_2=-SVx_1x_2Vx_1x_2. \tag{6}$$

But we know that $\lambda a = Vx_1x_2$, and hence we find

$$\Delta a = Vx_1x_2. \tag{7}$$

By multiplication we find also

$$\Delta = -Sax_1x_2. \tag{8}$$

In like manner we write $\Delta_0 a = Va_1a_2$ and find

$$\Delta_0 = -Saa_1a_2. \tag{9}$$

But we know that the Gaussian curvature k is given by

$$\begin{aligned}\Delta k &= \frac{1}{\Delta}(LN-M^2)\\ &= \frac{1}{\Delta}(Sa_1x_1Sa_2x_2 - Sa_1x_2Sa_2x_1),\ \text{since } Sa_2x_1 = Sa_1x_2 = M,\\ &= -\frac{1}{\Delta}SVa_1a_2Vx_1x_2 = -Saa_1a_2. \end{aligned} \tag{10}$$

$$\therefore\ \Delta k = \Delta_0. \tag{11}$$

By the use of r and w defined above we have the quadratic form:

$$\begin{aligned}\theta &= E\phi_2^2 - 2F\phi_2\phi_1 + G\phi_1^2 = -S(\phi_2x_1-\phi_1x_2)(\phi_2x_1-\phi_1x_2) = -Sww\\ &= -r^2Sx'x',\ \text{since } w=\phi_2x_1-\phi_1x_2 = rx',\\ &= \ r^2,\ \text{since } Sx'x' = -1. \end{aligned} \tag{12}$$

Also we have the quadratic:

$$\begin{aligned}\chi &= L\phi_2^2 - 2M\phi_2\phi_1 + N\phi_1^2 = S(\phi_2x_1-\phi_1x_2)(\phi_2a_1-\phi_1a_2) = Sw\overline{\omega}\\ &= r^2Sa'x',\ \text{since } ra' = \overline{\omega} \text{ and } rx' = w,\\ &= r^2\frac{1}{\rho}, \end{aligned} \tag{13}$$

where $\frac{1}{\rho} = Sa'x'$ is the curvature of a normal section through the tangent at the point.

Many expressions may be abbreviated by the use of the operator $\frac{d}{dn}$ where dn is an element of the arc of the curve on the surface at right angles to the given curve.

At any point P of the curve on the surface we have x', the unit vector along the tangent to the curve, and a the unit vector normal to the surface. Let us take ξ' a unit vector in the tangent plane at right angles to x. Then we have

$$\xi' = \frac{dx}{dn} = Vax'.$$

Let us write $\xi' = cx_1 + ex_2$ where c and e are scalars to be determined. We find

$$Sax_1\xi' = eSax_1x_2, \qquad Sax_2\xi' = cSax_2x_1,$$
$$\therefore \ \Delta e = -Sax_1\xi' = -Sax_1Vax' = -Sx_1x', \qquad \therefore \ \Delta c = Sax_2\xi' = Sax_2Vax' = Sx_2x'.$$

Hence we have

$$\xi' = -\frac{1}{\Delta}\{x_2Sx_1x' - x_1Sx_2x'\}, \text{ or } \frac{dx}{dn} = -\frac{1}{\Delta}\left\{\frac{\partial x}{\partial v}Sx_1x' - \frac{\partial x}{\partial u}Sx_2x'\right\}.$$

If now R be any function of u and v we find

$$\frac{dR}{dn} = \frac{dR}{dx}\cdot\frac{dx}{dn} = -\frac{1}{\Delta}\left\{\frac{\partial R}{\partial v}Sx_1x' - \frac{\partial R}{\partial u}Sx_2x'\right\}. \tag{14}$$

The operator $\frac{d}{dn}$ is very useful in what follows. In particular we have

$$\Delta\frac{d\phi}{dn} = -\{\phi_2Sx'x_1 - \phi_1Sx'x_2\} = -\{Sx'(\phi_2x_1 - \phi_1x_2)\} = -Sx'w = r,$$

$$\therefore \ \frac{d\phi}{dn} = \frac{r}{\Delta} \equiv \beta, \tag{15}$$

where we define the quantity β by $\beta = \frac{r}{\Delta}$.

From a study of curves on surfaces we find

$$D = -Sax'x'' \tag{16}$$

to be the *Geodesic curvature*, and the expression

$$W = -Saa'x' \tag{17}$$

to be the *Geodesic torsion*.

Also we have the cubic

$$K = Sa''x' - Sa'x'' = Pu'^3 + 3Qu'^2v' + 3Ru'v'^2 + Sv'^3, \tag{18}$$

where

$$\left.\begin{aligned} &P = Sa_{11}x_1 - Sx_{11}a_1, \quad Q = Sa_{12}x_1 - Sx_{12}a_1 = Sa_{11}x_2 - Sx_{11}a_2, \\ &R = Sa_{12}x_2 - Sx_{12}a_2 = Sa_{22}x_1 - Sx_{22}a_1, \quad S = Sa_{22}x_2 - Sx_{22}a_2. \end{aligned}\right\} \tag{19}$$

The mean curvature h is given by

$$\begin{aligned} \Delta h &= \frac{1}{\Delta}\{EN - 2FM + GL\} \\ &= -\frac{1}{\Delta}\{(Sx_1x_1Sa_2x_2 - Sx_1x_2Sa_2x_1) - (Sx_1x_2Sa_1x_2 - Sx_2x_2Sa_1x_1)\} \\ &= \frac{1}{\Delta}\{SVx_1x_2Va_1x_2 - SVx_1x_2Va_2x_1\} = Saa_1x_2 - Saa_2x_1. \end{aligned} \tag{20}$$

From this expression for h we easily find

$$\left.\begin{aligned}\Delta^2 h_1 &= ER-2FQ+GP, \qquad \Delta^2 h_2 = ES-2FR+GQ,\\ \Delta^2 h' &= E(Ru'+Sv')-2F(Qu'+Rv')+G(Pu'+Qv').\end{aligned}\right\} \tag{21}$$

Change of Parameters.—Suppose we have the surface $\bar{x}=f(\bar{u}\bar{v})$ and wish to change to the parameters u, v, where $u=P(\bar{u}\bar{v})$, $v=Q(\bar{u}\bar{v})$. Let us write,

$$\frac{\partial u}{\partial \bar{u}}=\frac{\partial P}{\partial \bar{u}}=P_1, \quad \frac{\partial u}{\partial \bar{v}}=\frac{\partial P}{\partial \bar{v}}=P_2, \quad \frac{\partial v}{\partial \bar{u}}=\frac{\partial Q}{\partial \bar{u}}=Q_1, \quad \frac{\partial v}{\partial \bar{v}}=\frac{\partial Q}{\partial \bar{v}}=Q_2.$$

Then we have the transformation

$$\bar{x}_1=x_1P_1+x_2Q_1, \quad \bar{x}_2=x_1P_2+x_2Q_2. \tag{22}$$

The modulus of this transformation is denoted by

$$\delta=P_1Q_2-P_2Q_1=\begin{vmatrix} P_1 & P_2 \\ Q_1 & Q_2 \end{vmatrix}. \tag{23}$$

We wish now to consider forms of expression that are invariant in transformation. When the resulting expression has the same form multiplied by a power of δ it is said to be a *relative* invariant form. When it has the same form, but not multiplied by a power of δ it is said to be an *absolute* invariant.

Elementary Invariant Vector Forms.—When a vector $\bar{\xi}$ is transformed into the vector ξ we have

Form (I) $\quad d\bar{\xi}=\bar{\xi}_1 d\bar{u}+\bar{\xi}_2 d\bar{v}=(\xi_1P_1+\xi_2Q_1)d\bar{u}+(\xi_1P_2+\xi_2Q_2)d\bar{v}$

$$=\xi_1(P_1d\bar{u}+P_2d\bar{v})+\xi_2(Q_1d\bar{u}+Q_2d\bar{v})=\xi_1du+\xi_2dv=d\xi.$$

This shows the complete differential to be invariant, and hence we have the complete derivative to be invariant. Hence vector expressions composed of factors that are complete derivatives are invariant. Thus

$$D=-Sax'x'', \quad W=-Saa'x', \quad Sx'x', \text{ \&c.},$$

are invariant.

Form (II) $\quad V\bar{\xi}_1\bar{\xi}_2=V(\xi_1P_1+\xi_2Q_1)(\xi_1P_2+\xi_2Q_2)$

$$=(P_1Q_2-P_2Q_1)V\xi_1\xi_2=\delta V\xi_1\xi_2,$$

which shows this form to be invariant.

Under this form we have the invariants

$$\Delta=-Sax_1x_2=-SaVx_1x_2, \quad \Delta k=-Saa_1a_2=-SaVa_1a_2, \quad Saw_1w_2=SaVw_1w_2.$$

Form (III) $\quad S\bar{\xi}_1\bar{x}_2-S\bar{\xi}_2\bar{x}_1=S(\xi_1P_1+\xi_2Q_1)(x_1P_2+x_2Q_2)$

$$-S(\xi_1P_2+\xi_2Q_2)(x_1P_1+x_2Q_1)=(P_1Q_2-P_2Q_1)(S\xi_1x_2-S\xi_2x_1)$$
$$=\delta(S\xi_1x_2-S\xi_2x_1).$$

This form is made up of the difference of two scalar terms of which the partial derivatives 1, 2 in the first term are respectively 2, 1 in the second term.

In a similar manner we may show the 1, 2 and 2, 1 form to apply to a large number of invariant expressions. Thus,

$$Sx_1w_2 - Sx_2w_1, \quad Sax_1w_2 - Sax_2w_1, \quad x_1\Delta_2 - x_2\Delta_1, \quad \phi_2x_1 - \phi_1x_2, \text{ \&c.},$$

are all invariant forms.

Of especial importance under Form (III) is the invariant expression

$$\frac{dK}{dn} = -\frac{1}{\Delta}\{K_2Sx'x_1 - K_1Sx'x_2\}, \quad \text{see (14)}$$

where K represents either a scalar or a vector quantity. By the use of this invariant expression we have, when K is a vector, a number of invariant forms. Thus each factor in

$$Sa\frac{dK}{dn}, \quad Sax'\frac{dK}{dn}, \quad Saw\frac{dK}{dn}, \quad Sx''\frac{dK}{dn}, \quad Sax'\frac{dK'}{dn}, \quad Saw\frac{dK'}{dn},$$

being invariant, the forms themselves are invariant.

Every invariant in differential geometry may be shown to be composed of one or more of the elementary forms (I), (II), or (III). If we can evaluate any of the invariant forms that we can write out from our simple elements in terms of well-known invariant expressions, we will be enabled to evaluate any invariant or covariant expression of differential geometry.

Forsyth has shown in his "Differential Geometry" that all invariants and covariants formed by the use of derivatives below the third order may be expressed in terms of any set of twelve such expressions that may be selected. From the set selected by him he derived eleven absolute invariant expressions as his fundamental set. For the set that we will use in this paper let us take the following by means of which we will evaluate all other invariant forms made by derivatives below the third order:

$D = -Sax'x''$, the geodesic curvature.

$W = -Saa'x'$, the geodesic torsion.

$k = -\dfrac{1}{\Delta}Saa_1a_2$, the Gaussian curvature where $\Delta = -Sax_1x_2$.

$\beta = \dfrac{r}{\Delta} = \dfrac{\sqrt{-Sww}}{\Delta}$, the differential parameter of the first order.

$h = \dfrac{1}{\Delta}\{Saa_1x_2 - Saa_2x_1\}$, the mean curvature.

$\dfrac{1}{\rho} = Sa'x' = -Sax''$, the curvature of a normal section.

$$\frac{d\beta}{ds}, \quad \frac{d\beta}{dn}, \quad \frac{dh}{ds}, \quad \frac{dh}{dn}, \quad \frac{d}{ds}\left(\frac{1}{\rho}\right), \quad \frac{d}{dn}\left(\frac{1}{\rho}\right).$$

Between these there exists the relation

$$k=\frac{h}{\rho}-\frac{1}{\rho^2}-W^2,$$

making in reality only eleven absolute invariant forms to be used.

We will now evaluate a number of *invariant forms* and will later show that invariants of differential geometry are composed of those forms. It will be seen that every form that we notice will be invariant by reason of elementary Form (I), Form (II) or Form (III). Hence, while we may evaluate invariant or covariant expressions of differential geometry by means of a set of eleven, we may express all such invariants or covariants in terms of *three elementary type forms.*

II. *Evaluation of Invariant Forms.*

We have defined

$$w=\phi_2x_1-\phi_1x_2=rx', \tag{24}$$

which is seen to be invariant by type form III. Also we have

$$Sww=r^2Sx'x'=-r^2, \text{ since } Sx'x'=-1. \tag{25}$$

$$Sx'w'=Sx'(r'x'+rx'')=-r', \text{ since } Sx'x''=0. \tag{26}$$

$$Sww'=rSx'w'=-rr'. \tag{27}$$

Since $Sax_1x_2=-\Delta$ we have

$$Sawx_2=Sa(\phi_2x_1-\phi_1x_2)x_2=-\phi_2\Delta, \quad Sawx_1=Sa(\phi_2x_1-\phi_1x_2)x_1=-\phi_1\Delta.$$

By differentiation we find

$$Saw_1x_2+Sawx_{12}=-\phi_{12}\Delta-\phi_2\Delta_1, \quad Saw_2x_1+Sawx_{12}=-\phi_{12}\Delta-\phi_1\Delta_2.$$

$$\therefore\ Saw_2x_1-Saw_1x_2=\phi_2\Delta_1-\phi_1\Delta_2=r\Delta'. \tag{28}$$

We have seen that we may write $ru'=\phi_2$ and $rv'=-\phi_1$, and from these we find $r_1u'+ru''=\phi_{12}=-r_2v'-rv''$,

$$\therefore\ r(u''+v'')=-\{r_1u'+r_2v'\}=-r'. \tag{29}$$

We have seen that

$$\Delta h=\ Saa_1x_2-Saa_2x_1. \text{ See (20)} \tag{30}$$

$$\Delta k=-Saa_1a_2. \text{ See (10)}. \tag{31}$$

$$\begin{aligned}Sa'a'&=-\frac{1}{\Delta}Sa'a'Sax_1x_2\\&=-\frac{1}{\Delta}\{Sa'x_1Saa'x_2-Sa'x_2Saa'x_1\}, \text{ since } Saa'=0\\&=-\frac{1}{\Delta}\{Sa_1x'Saa'x_2-Sa_2x'Saa'x_1\}, \text{ since } Sa'x_1=Sa_1x', \text{ \&c.,}\end{aligned}$$

$$=-\frac{1}{\Delta}\{Sa'x'(Saa_1x_2-Saa_2x_1)+Sx'x_2Saa'a_1-Sx'x_1Saa'a_2\}$$

$$=-\frac{1}{\Delta}\left\{\frac{\Delta h}{\rho}-Saa_1a_2Sx'(x_2v'+x_1u')\right\},\ \text{since } Saa'a_1=v'Saa_2a_1$$

$$=-\frac{1}{\Delta}\left\{\frac{\Delta h}{\rho}+\Delta kSx'x'\right\}=-\frac{h}{\rho}+k. \tag{32}$$

It is easily seen that the vector $Va'x'$ is parallel to a, the unit vector normal to the surface. Hence we have

$$Wa=Va'x',\ \text{since } W=-Saa'x'.$$

$$\therefore\ W^2=-SVa'x'Va'x'=-Sa'x'Sa'x'+Sa'a'Sx'x'=-\frac{1}{\rho^2}+\frac{h}{\rho}-k. \tag{33}$$

$$WD=Saa'x'Sax'x''$$

$$=-\begin{vmatrix} Saa & Sax' & Sax'' \\ Sa'a & Sa'x' & Sa'x'' \\ Sx'a & Sx'x' & Sx'x'' \end{vmatrix}=-\begin{vmatrix} -1 & 0 & Sax'' \\ 0 & Sa'x' & Sa'x'' \\ 0 & -1 & 0 \end{vmatrix},\ \text{since } Sx'x''=0$$

$$=Sa'x''. \tag{34}$$

$$Saa'x''=-Sx'x'Saa'x''=-Sx'a'Sax'x''-Sx'x''Saa'x',\ \text{since } Sx'a=0$$

$$=\frac{D}{\rho},\ \text{since } Sx'x''=0. \tag{35}$$

$$Saa'w'=Saa'(r'x'+rx'')=-r'W+\frac{rD}{\rho}. \tag{36}$$

$$Sx'\frac{dx}{dn}=Sx'Vax'=0. \tag{37}$$

$$Sx'\frac{dw}{dn}=Sx'\left(r\frac{dx'}{dn}+x'\frac{dr}{dn}\right),\ \text{since } w=rx'$$

$$=-\frac{dr}{dn}. \tag{38}$$

$$Sx'\frac{da}{dn}=-\frac{1}{\Delta}\{Sx'x_1Sx'a_2-Sx'x_2Sx'a_1\},\ \text{see (14)}$$

$$=-\frac{1}{\Delta}\{Sx'x_1Sa'x_2-Sx'x_2Sa'x_1\},\ \text{since } Sa'x_2=Sx'a_2 \text{ and } Sa'x_1=Sx'a_1$$

$$=-\frac{1}{\Delta}SVx_1x_2Va'x'=-Saa'x',\ \text{since } \Delta a=Vx_1x_2$$

$$=W. \tag{39}$$

$$Sa\frac{dw}{dn}=-Sw\frac{da}{dn},\ \text{since } Saw=0$$

$$=-rSx'\frac{da}{dn}=-rW. \tag{40}$$

$$Sa'\frac{dx}{dn} = \quad Sa'Vax' = -Saa'x' = W. \tag{41}$$

$$\begin{aligned} Sax'\frac{dw}{dn} &= -\frac{1}{\Delta}\{Sx'x_1Sax'w_2 - Sx'x_2Sax'w_1\} \\ &= -\frac{1}{\Delta}\{Sx'x'(Sax_1w_2 - Sax_2w_1) + Sx'w_2Sax'x_1 - Sx'w_1Sax'x_2\} \\ &= -\frac{1}{\Delta}\{r\Delta' + \Delta Sx'(w_2v' + w_1u')\} = -\frac{1}{\Delta}(r\Delta' + \Delta Sx'w') \\ &= -\frac{1}{\Delta}(r\Delta' - \Delta r') = \frac{\Delta r' - r\Delta'}{\Delta} = \Delta\left(\frac{r}{\Delta}\right)' = \Delta\beta', \end{aligned}$$

$$\text{since we define } \beta = \frac{r}{\Delta}. \tag{42}$$

$$Saw\frac{dw}{dn} = rSax'\frac{dw}{dn} = r\Delta\beta', \text{ since } w = rx'. \tag{43}$$

$$\begin{aligned} Sax'\frac{dx'}{dn} &= Sax'\frac{d}{dn}\left(\frac{w}{r}\right) = \frac{1}{r}Sax'\frac{dw}{dn}, \text{ since } \frac{d}{du}\left(\frac{w}{r}\right) = \frac{1}{r}\frac{dw}{dn} + w\frac{d}{dn}\left(\frac{1}{r}\right) \\ &= \frac{1}{r}\Delta\beta' = \frac{\beta'}{\beta}. \end{aligned} \tag{44}$$

$$Sax'\frac{dx}{dn} = SVax'Vax' = -SaaSx'x' = -1. \tag{45}$$

$$Saw'\frac{dx}{dn} = -Sx'x'Saw'\frac{dx}{dn} = -Sx'w'Sax'\frac{dx}{dn} = -r'. \tag{46}$$

$$\begin{aligned} Saw'\frac{dw}{dn} &= -Sx'x'Saw'\frac{dw}{dn} = -Sx'w'Sax'\frac{dw}{dn} - Sx'\frac{dw}{dn}Saw'x' \\ &= +r'\Delta\beta' - rDSx'\frac{d}{dn}(rx'), \text{ since } Sax'w' = Sax'(r'x' + rx'') = -rD \\ &= +r'\Delta\beta' + rD\frac{dr}{dn}. \end{aligned} \tag{47}$$

$$Saw'\frac{dx'}{dn} = -Sx'x'Saw'\frac{dx'}{dn} = -Sx'w'Sax'\frac{dx'}{dn} - Sx'\frac{dx'}{dn}Saw'x' = \frac{\beta'}{\beta}\frac{r'}{1}. \tag{48}$$

$$Saa'\frac{dx'}{dn} = -Sx'x'Saa'\frac{dx'}{dn} = -Sx'a'Sax'\frac{dx'}{dn} - Sx'\frac{dx'}{dn}Saa'x' = -\frac{1}{\rho}\frac{\beta'}{\beta}. \tag{49}$$

$$\begin{aligned} Saa'\frac{dw}{dn} &= -Sx'x'Saa'\frac{dw}{dn} \\ &= -Sx'a'Sax'\frac{dw}{dn} - Sx'\frac{dw}{dn}Saa'x' = -\frac{1}{\rho}\Delta\beta' - W\frac{dr}{dn}. \end{aligned} \tag{50}$$

$$Saa'\frac{dx}{dn}=SVaa'Vax'=-SaaSa'x'=\frac{1}{\rho}. \tag{51}$$

$$\begin{aligned}Sax'\frac{da}{dn}&=-\frac{1}{\Delta}\{Sx'x_1Sax'a_2-Sx'x_2Sax'a_1\}\\&=-\frac{1}{\Delta}\{Sx'x'(Sax_1a_2-Sax_2a_1)+Sx'a_2Sax'x_1-Sx'a_1Sax'x_2\}\\&=-\frac{1}{\Delta}\{-\Delta h+\Delta Sx'(a_2v'+a_1u')\}=h-\frac{1}{\rho}.\end{aligned} \tag{52}$$

$$\begin{aligned}Saw'\frac{da}{dn}&=-Sx'x'Saw'\frac{da}{dn}\\&=-Sx'w'Sax'\frac{da}{dn}-Sx'\frac{da}{dn}Saw'x'=r'\left(h-\frac{1}{\rho}\right)-rDW.\end{aligned} \tag{53}$$

$$\begin{aligned}Saa'\frac{da}{dn}&=-Sx'x'Saa'\frac{da}{dn}=-Sx'a'Sax'\frac{da}{dn}-Sx'\frac{da}{dn}Saa'x'\\&=-\frac{1}{\rho}\left(h-\frac{1}{\rho}\right)+W^2=-k.\quad\text{See (33).}\end{aligned} \tag{54}$$

$$Sax''\frac{da}{dn}=-Sx'x'Sax''\frac{da}{dn}=-Sx'\frac{da}{dn}Sax''x'=-WD. \tag{55}$$

$$\begin{aligned}Saw_1w_2&=-Sx'x'Saw_1w_2=-Sx'(x_1u'+x_2v')Saw_1w_2\\&=-\{Sx'x_1Saw'w_2-Sx'x_2Saw'w_1\}\\&=\quad\Delta Saw'\frac{dw}{dn}=\Delta^2r'\beta'+\Delta rD\frac{dr}{dn}.\quad\text{See (47).}\end{aligned} \tag{56}$$

$$\begin{aligned}Sx''\frac{da}{dn}&=-\frac{1}{W}Sx''\frac{da}{dn}Saa'x'\\&=+\frac{1}{W}Sx''a'Sax'\frac{da}{dn},\ \text{since } Sa'x'\frac{da}{dn}=0 \text{ and } Sx''x'=0\\&=\quad D\left(h-\frac{1}{\rho}\right).\quad\text{See (34).}\end{aligned} \tag{57}$$

$$\begin{aligned}Sa'\frac{dw}{dn}&=-\frac{1}{rD}Sa'\frac{dw}{dn}Sax'w'=-\frac{1}{rD}\left\{-Sa'x'Saw'\frac{dw}{dn}+Sa'w'Sax'\frac{dw}{dn}\right\}\\&=-\frac{1}{rD}\left\{-\frac{1}{\rho}\left(r'\Delta\beta'+rD\frac{dr}{du}\right)+\Delta\beta'\left(\frac{r'}{\rho}+rDW\right)\right\}\\&=-\frac{1}{rD}\left\{-rD\frac{dr}{du}+rDW\Delta\beta'\right\}=\frac{dr}{du}-\Delta W\beta'.\end{aligned} \tag{58}$$

$$\begin{aligned}Sa'\frac{dx'}{dn}&=-\frac{1}{rD}Sa'\frac{dx'}{dn}Sax'w'=-\frac{1}{rD}\left\{-Sa'x'Saw'\frac{dx'}{dn}+Sa'w'Sax'\frac{dx'}{dn}\right\}\\&=-\frac{1}{rD}\left\{-\frac{1}{\rho}\frac{\beta'}{\beta}r'+\frac{\beta'}{\beta}\left(\frac{r'}{\rho}+rWD\right)\right\}=-W\frac{\beta'}{\beta}.\end{aligned} \tag{59}$$

$$Sw\frac{da'}{dn}=\frac{d}{dn}\left(\frac{r}{\rho}\right)-Sa'\frac{dw}{dn},\ \text{since }\ Sa'w=\frac{r}{\rho}$$

$$=\frac{d}{dn}\left(\frac{r}{\rho}\right)-Sa'\left(r\frac{dx'}{dn}+x'\frac{dr}{dn}\right),\ \text{since }\ w=rx'$$

$$=\frac{d}{dn}\left(\frac{r}{\rho}\right)+rW\frac{\beta'}{\beta}-\frac{1}{\rho}\frac{dr}{dn}$$

$$=r\frac{d}{dn}\left(\frac{1}{\rho}\right)+\frac{1}{\rho}\frac{dr}{dn}+rW\frac{\beta'}{\beta}-\frac{1}{\rho}\frac{dr}{dn}=r\frac{d}{dn}\left(\frac{1}{\rho}\right)+rW\frac{\beta'}{\beta}. \tag{60}$$

$$Sx'\frac{da'}{dn}=\frac{d}{dn}\left(\frac{1}{\rho}\right)+\frac{\Delta W\beta'}{r}. \tag{61}$$

$$Sx''\frac{dw}{dn}=-\frac{1}{W}Sx''\frac{dw}{dn}Saa'x'=-\frac{1}{W}\left\{Sx''aSa'x'\frac{dw}{dn}-Sx''a'Sax'\frac{dw}{dn}\right\}$$

$$=-\frac{1}{W}\left\{-\frac{1}{\rho}WSa\frac{dw}{dn}-WD\Delta\beta'\right\},\ \text{since }\ Wa=Va'x'$$

$$=-\frac{r}{\rho}W+D\Delta\beta'. \tag{62}$$

$$Sa''\frac{dx}{dn}=-Saa''x',\ \text{since }\frac{dx}{dn}=Vax'$$

$$=-Saa''x'-Saa'x''+Saa'x''=W'+\frac{D}{\rho}.\quad \text{See (35).} \tag{63}$$

$$Sa''\frac{dx}{dn}=\frac{1}{\Delta}\{Sx'x_2Sa''x_1-Sx'x_1Sa''x_2\}$$

$$=\frac{1}{\Delta}\{Sx'x_2(Sa''x_1-Sx''a_1)-Sx'x_1(Sa''x_2-Sx''a_2)\}+Sx''\frac{da}{dn}$$

$$=-\frac{1}{\Delta}[Sx'x_2\{Sa'(x_1)'-Sx'(a_1)'\}-Sx'x_1\{Sa'(x_2)'-Sx'(a_2)'\}]$$

$$+Sx''\frac{da}{dn},\ \text{since }\ Sx'a_1=Sa'x_1$$

$$=-\frac{1}{\Delta}[Sx'x_2\{Sa'(x')_1-Sx'(a')_1\}-Sx'x_1\{Sa'(x')_2-Sx'(a')_2\}]$$

$$+Sx''\frac{da}{dn}$$

$$=\frac{1}{\Delta}[Sx'x_2\{Sa'(x')_1+Sx'(a')_1\}-Sx'x_1\{Sa'(x')_2+Sx'(a')_2\}]$$

$$-\frac{2}{\Delta}[Sx'x_2Sa'(x')_1-Sx'x_1Sa'(x')_2]+Sx''\frac{da}{dn}=\frac{d}{dn}\left(\frac{1}{\rho}\right)$$

$$-\frac{2}{\Delta}Sa'\frac{dx'}{dn}+Sx''\frac{da}{dn}=\frac{d}{dn}\left(\frac{1}{\rho}\right)+2W\frac{\beta'}{\beta}+D\left(h-\frac{1}{\rho}\right). \tag{64}$$

See (59) and (57).

$$W' = -Saa''x' - Saa'x'' = Sa''Vax' - Saa'x'' = Sa''\frac{dx}{dn} - Saa'x''$$
$$= \frac{d}{dn}\left(\frac{1}{\rho}\right) + 2W\frac{\beta'}{\beta} + D\left(h - \frac{1}{\rho}\right) - \frac{D}{\rho}. \quad \text{See (64) and (35).}$$
$$= \frac{d}{dn}\left(\frac{1}{\rho}\right) + 2W\frac{\beta'}{\beta} + D\left(h - \frac{2}{\rho}\right). \tag{65}$$

$$Saa'a'' = -Sx'x'Saa'a'' = -Sx'a'Sax'a'' - Sx'a''Saa'x'$$
$$= -\frac{1}{\rho}(Sax''a' + Sax'a'') + \frac{1}{\rho}Sax''a' + WSa''x'$$
$$= -\frac{W'}{\rho} - \frac{D}{\rho^2} + W(K + WD), \text{ since } K = Sa''x' - Sa'x'' = Sa''x' - WD$$
$$= -\frac{1}{\rho}\frac{d}{dn}\left(\frac{1}{\rho}\right) - D\left(\frac{h}{\rho} - \frac{1}{\rho^2} - W^2\right) - \frac{2}{\rho}\frac{W\beta'}{\beta} + WK$$
$$= -\frac{1}{\rho}\frac{d}{dn}\left(\frac{1}{\rho}\right) - Dk - \frac{2W\beta'}{\beta} + WK. \tag{66}$$

By the use of similar methods we find

$$Sx_2w_1 - Sx_1w_2 = -\Delta\left(rD - \frac{dr}{dn}\right). \tag{67}$$

$$Sx'(\Delta_1 a_2 - \Delta_2 a_1) = \Delta\left(W\Delta' - \frac{1}{\rho}\frac{d\Delta}{dn}\right). \tag{68}$$

$$Saa_1w_2 - Saa_2w_1 = -\Delta W\left(rD + \frac{dr}{dn}\right) - \frac{\Delta}{\rho}\Delta'\beta - \frac{2\Delta^2\beta'}{\rho} + r'\Delta h. \tag{69}$$

$$Sax'(\Delta_1 w_2 - \Delta_2 w_1) = -r\Delta'^2 + \Delta rD\frac{d\Delta}{dn} + \Delta\Delta' r'. \tag{70}$$

$$Sax'(\Delta_1 a_2 - \Delta_2 a_1) = \Delta\Delta' h - \Delta W\frac{d\Delta}{dn} - \frac{\Delta}{\rho}\Delta'. \tag{71}$$

$$Sa_1w_2 - Sa_2w_1 = -\Delta rD\left(h - \frac{1}{\rho}\right) + \frac{\Delta}{\rho}\frac{dr}{dn} - 2\Delta^2 W\beta' - r\Delta' W. \tag{72}$$

$$Saa_1(x')_2 - Saa_2(x')_1 = -\frac{\Delta\beta'}{\rho\beta} - \Delta WD. \tag{73}$$

$$Sa(a')_1x_2 - Sa(a')_2x_1 = \Delta h' - \frac{\Delta\beta'}{\beta}\left(h - \frac{1}{\rho}\right) + \Delta WD. \tag{74}$$

Each form evaluated above is seen to be invariant by virtue of the type forms I, II or III, of which it is composed.

III. *Invariants of Differential Geometry.*

In what follows invariants or covariants of differential geometry will be seen to be composed of one or more invariant forms, and hence may be expressed in terms of one or more of the three type forms I, II and III. It will thus be seen, also, that the invariants and covariants may be evaluated by the use of the results obtained in the previous section.

In making the evaluations of invariant forms the derivatives of r and Δ were used in several instances. The reader will notice that in combining two or more of these forms to express an invariant or covariant in differential geometry that the derivatives of r and Δ are eliminated by the use of the relation $r=\Delta\beta$.

In what follows the invariancy of an expression will be indicated by giving at the right the type form or forms of which it is composed.

The quadratic form

$$\theta=E\phi_2^2-2F\phi_2\phi_1+G\phi_1^2$$

was shown in (25) to be given by

$$\theta=-Sww=-S(\phi_2x_1-\phi_1x_2)(\phi_2x_1-\phi_1x_2) \quad \text{(Form III).} \qquad \text{(a)}$$

From this we obtain the absolute invariant

$$\frac{\theta}{\Delta^2}=-\frac{Sww}{\Delta^2}=\beta^2, \qquad \text{(a}'\text{)}$$

where β^2 is the differential parameter of the first order.

Again we have the quadratic form

$$\begin{aligned}\chi&=L\phi_2^2-2M\phi_2\phi_1+N\phi_1^2=S(\phi_2x_1-\phi_1x_2)(\phi_2a_1-\phi_1a_2) \quad \text{(Form III).}\\ &=Sw\overline{\omega}, \text{ since } \overline{\omega}=\phi_2a_1-\phi_1a_2\\ &=r^2Sx'a', \text{ since } w=rx' \text{ and } \overline{\omega}=ra'=\frac{r^2}{\rho}. \quad \text{See (13).} \qquad \text{(b)}\end{aligned}$$

From this we obtain the absolute invariant form

$$\frac{\chi}{\Delta^2}=\frac{\beta^2}{\rho}, \text{ since } \beta=\frac{r}{\Delta}. \qquad \text{(b}'\text{)}$$

We have seen also in (10),

$$\begin{aligned}k&=\frac{1}{\Delta^2}(LN-M^2)=\frac{1}{\Delta^2}(Sa_1x_1Sa_2x_1-Sa_2x_1Sa_1x_2)\\ &=-\frac{1}{\Delta^2}SVa_1a_2Vx_1x_2 \quad \text{(Form II)}\\ &=-\frac{1}{\Delta}Saa_1a_2\,.. \qquad \text{(c)}\end{aligned}$$

In differential geometry this absolute invariant is often written

$$k=\frac{1}{\rho_1\rho_2}. \tag{c'}$$

After omitting the numerical factors the functional determinant of θ and χ may be written

$$J_{\theta\chi}=-\begin{vmatrix} Sx_1w, & Sx_2w \\ Sa_1w, & Sa_2w \end{vmatrix}=-r^2\{Sx_1x'Sx'a_2-Sx_2x'Sx'a_1\}$$

$$= \quad \Delta r^2Sx'\frac{da}{dn}=\Delta r^2W. \quad \text{See (39).} \tag{d}$$

From this we find the absolute invariant

$$\frac{J_{\theta\chi}}{\Delta^3}=\beta^2W. \tag{d'}$$

We have the quadratic

$$d=A\phi_2^2-2B\phi_2\phi_1+C\phi_1^2=r^2(Au'^2+2Bu'v'+Cv^2),$$

where

$$\Delta A=\Delta\phi_{11}+Sawx_{11}, \quad \Delta B=\Delta\phi_{12}+Sawx_{12}, \quad \Delta C=\Delta\phi_{22}+Sawx_{22}.$$

$$\therefore\; d=\frac{r^2}{\Delta}\{\Delta(\phi_{11}u'^2+2\phi_{12}u'v'+\phi_{22}v'^2)+Saw(x_{11}u'^2+2x_{12}u'v'+x_{22}v'^2)\}$$

$$=\frac{r^2}{\Delta}\{-\Delta(\phi_1u''+\phi_2v'')+Saw(x''-x_1u''-x_2v'')\}$$

$$=\frac{r^2}{\Delta}\{-\Delta(\phi_1u''+\phi_2v'')+Sawx''+\Delta(\phi_1u''+\phi_2v'')\}$$

$$=\frac{r^3}{\Delta}Sax'x'' \quad \text{(Form I).} \tag{e}$$

In this last form d is seen to be invariant. We have also the absolute invariant form

$$\frac{d}{\Delta^2}=-\beta^3D, \text{ since } D=-Sax'x''. \tag{e'}$$

More useful expressions for A, B and C may be found as follows:

$$Sawx_1=Sa(\phi_2x_1-\phi_1x_2)x_1=-\phi_1\Delta, \quad Sawx_2=Sa(\phi_2x_1-\phi_1x_2)x_2=-\phi_2\Delta.$$

By differentiation we have,

$$Sawx_{11}+Saw_1x_1=-\phi_1\Delta_1-\phi_{11}\Delta, \quad Sawx_{12}+Saw_2x_1=-\phi_1\Delta_2-\phi_{12}\Delta,$$
$$Sawx_{12}+Saw_1x_2=-\phi_2\Delta_1-\phi_{12}\Delta, \quad Sawx_{22}+Saw_2x_2=-\phi_2\Delta_2-\phi_{22}\Delta,$$

and by the use of these relations we find

$$\Delta A=-\phi_1\Delta_1-Saw_1x_1, \quad \Delta B=-\phi_1\Delta_2-Saw_2x_1=-\phi_2\Delta_1-Saw_1x_2,$$
$$\Delta C=-\phi_2\Delta_2-Saw_2x_2.$$

The intermediate invariant of θ and d may be written

$$I=AG-2BF+CE=\frac{1}{\Delta}\{Sx_2x_2(\Delta_1\phi_1+Saw_1x_1)-Sx_1x_2(\phi_2\Delta_1+Saw_1x_2)$$
$$-Sx_1x_2(\phi_1\Delta_2+Saw_2x_1)+Sx_1x_1(\phi_2\Delta_2+Saw_2x_2)\}$$
$$=\frac{1}{\Delta}\{\Delta_1Sx_2(\phi_1x_2-\phi_2x_1)-\Delta_2Sx_1(\phi_1x_2-\phi_2x_1)+Sx_2w_1Sax_2x_1$$
$$+Sx_2x_1Saw_1x_2-Sx_1x_2Saw_1x_2-Sx_1w_2Sax_2x_1+Sx_1x_1Saw_2x_2-Sx_1x_1Saw_2x_2\}$$
$$=\frac{1}{\Delta}\{\Delta_2Sx_1w-\Delta_1Sx_2w+\Delta(Sx_2w_1-Sx_1w_2)\}\quad\text{(Form III)}$$
$$=\frac{1}{\Delta}\left\{-r\Delta\frac{d\Delta}{dn}-\Delta^2\left(rD-\frac{dr}{dn}\right)\right\}.\quad\text{See (14) and (67)}$$
$$=-\Delta rD+\Delta\frac{dr}{dn}-\beta\frac{d\Delta}{dn}\Delta=\Delta^2\frac{d\beta}{dn}-\beta\Delta^2D,\ \text{since } r=\Delta\beta$$
$$=\ \Delta^2\left(\frac{d\beta}{dn}-\beta D\right),\tag{f}$$

and from this we have the absolute invariant

$$\frac{I}{\Delta^2}=\frac{d\beta}{dn}-\beta D.\tag{f'}$$

It is well to notice that the above invariant may be written

$$I=\Delta\left\{Sx_2\left(\frac{w}{\Delta}\right)_1-Sx_1\left(\frac{w}{\Delta}\right)_2\right\}\quad\text{(Form III)}$$

or $$\frac{I}{\Delta^2}=\frac{1}{\Delta}\left\{\left(\frac{Sx_2w}{\Delta}\right)_1-\left(\frac{Sx_1w}{\Delta}\right)_2\right\}=\frac{1}{\Delta}\left\{\frac{\partial}{\partial u}\left(\frac{G\phi_1-F\phi_2}{\Delta}\right)-\frac{\partial}{\partial v}\left(\frac{E\phi_2-F\phi_1}{\Delta}\right)\right\},$$

which is the *differential parameter of the second order.*

By omitting the numerical factors, the Jacobian of θ and d may be written:

$$J_{\theta d}=-\begin{vmatrix} Sx_1w, & Sx_2w \\ (A\phi_2-B\phi_1), & (B\phi_2-C\phi_1)\end{vmatrix}$$
$$=\frac{1}{\Delta}\begin{vmatrix} Sx_1w, & \phi_2(\phi_1\Delta_1+Saw_1x_1)-\phi_1(\phi_2\Delta_1+Saw_1x_2), \\ Sx_2w, & \phi_2(\phi_1\Delta_2+Saw_2x_1)-\phi_1(\phi_2\Delta_2+Saw_2x_2)\end{vmatrix}$$
$$=\frac{1}{\Delta}\begin{vmatrix} Sx_1w & Sx_2w \\ Saw_1w & Saw_2w\end{vmatrix},\ \text{since } w=\phi_2x_1-\phi_1x_2$$
$$=-\frac{r^2}{\Delta}\{Sx_1x'Sax'w_2-Sx_2x'Sax'w_1\}\quad\text{(Form III)}$$
$$=\ r^2Sax'\frac{dw}{du}=r^2\Delta\beta'.\quad\text{See (42).}\tag{g}$$
$$\therefore\ \frac{J\theta d}{\Delta^3}=\beta^2\beta'.\tag{g'}$$

The cubic

$$\overline{K}=P\phi_2^3-3Q\phi_2^2\phi_1+3R\phi_2\phi_1^2-S\phi_1^3=r^3(Pu'^3+3Qu'^2v'+3Ru'v'^2+Sv'^3)$$
$$=r^3(Sa''x'-Sa'x'')\quad\text{(Form I).}\tag{h}$$

is seen in this last form to be invariant.

This cubic may be expressed in terms of well-known invariants as follows:

$$\begin{aligned}\overline{K}&=r^3(Sa''x'+Sa'x''-2Sa'x'')\\&=r^3\left\{\frac{d}{ds}(Sa'x')-2Sa'x''\right\}=r^3\left\{\frac{d}{ds}\left(\frac{1}{\rho}\right)-2WD\right\},\end{aligned}$$

and from this we find

$$\frac{\overline{K}}{\Delta^3}=\beta^3\left\{\frac{d}{ds}\left(\frac{1}{\rho}\right)-2WD\right\}. \tag{h$'$}$$

From the cubic $\overline{K}$, let us take

$$\begin{aligned}K=Sa''x'-Sx''a'&=S(a_{11}u'^2+2a_{12}u'v'+a_{22}v'^2)x'-S(x_{11}u'^2+2x_{12}u'v'+x_{22}v'^2)a'\\&=u'^3\{Sa_{11}x_1-Sx_{11}a_1\}+u'^2v'\{(Sa_{11}x_2-Sx_{11}a_2)+2(Sa_{12}x_1-Sx_{12}a_1)\}\\&\quad+u'v'^2\{2(Sa_{12}x_2-Sx_{12}a_2)+(Sa_{22}x_1-Sx_{22}a_1)\}+v'^3\{Sa_{22}x_2-Sx_{22}a_2\}.\end{aligned}$$

Now we have $Sa_1x_2=Sx_1a_2$, and by differentiation we find

$$Sa_{11}x_2-Sx_{11}a_2=Sa_{12}x_1-Sx_{12}a_1,\quad Sa_{12}x_2-Sx_{12}a_2=Sa_{22}x_1-Sx_{22}a_1.$$

If now we write

$$K=Pu'^3+3Qu'^2v'+3Ru'v'^2+Sv'^3,$$

we have from the above,

$$\begin{aligned}P&=Sa_{11}x_1-Sx_{11}a_1,\quad Q=Sa_{12}x_1-Sx_{12}a_1=Sa_{11}x_2-Sx_{11}a_2,\\R&=Sa_{12}x_2-Sx_{12}a_2=Sa_{22}x_1-Sx_{22}a_1,\quad S=Sa_{22}x_2-Sx_{22}a_2.\end{aligned}$$

Let us now denote

$$K_1=Pu'^2+2Qu'v'+Rv'^2,\quad K_2=Qu'^2+2Ru'v'+Sv'^2.$$

Then we find

$$\begin{aligned}K_1&=u'(Pu'+Qv')+v'(Qu'+Rv')\\&=u'\{S(a_1)'x_1-S(x_1)'a_1\}+v'\{S(a_2)'x_1-S(x_2)'a_1\}\\&=u'\{S(a')_1x_1-S(x')_1a_1\}+v'\{S(a')_2x_1-S(x')_2a_1\}\\&=Sx_1\{(a')_1u'+(a')_2v'\}-Sa_1\{(x')_1u'+(x')_2v'\}\\&=Sa''x_1-Sx''a_1,\end{aligned}$$

and in like manner $\quad K_2=Sa''x_2-Sx''a_2.$

The expression

$$\begin{aligned}\sigma=&\quad\{E^2S-3EFR+(EG+2F^2)Q-FGP\}\phi_2\\&\qquad-\{EFS-(EG+2F^2)R+3FGQ-G^2P\}\phi_1\\=&\quad\{E(ES-2FR+GQ)-F(ER-2FQ+GP)\}\phi_2\\&\qquad-\{F(ES-2FR+GQ)-G(ER-2FQ+GP)\}\phi_1\\=&\quad\Delta^2\{(Eh_2-Fh_1)\phi_2-(Fh_2-Gh_1)\phi_1\}\quad\text{See (21)}\\=&-\Delta^2\{h_2Sx_1(\phi_2x_1-\phi_1x_2)-h_1Sx_2(\phi_2x_1-\phi_1x_2)\}\\=&-\Delta^2r\{h_2Sx_1x'-h_1Sx_2x'\}\quad\text{(Form III)}\\=&\quad\Delta^3r\frac{dh}{dn},\end{aligned}\tag{i}$$

is seen in the last two forms to be invariant.

From this we easily derive the absolute invariant form,

$$\frac{\sigma}{\Delta^4}=\beta\frac{dh}{dn}. \tag{i'}$$

Also we have

$$\begin{aligned}\sigma' &= (ER-2FQ-GP)\phi_2-(ES-2FR+GQ)\phi_1 \\ &= \Delta^2(h_1\phi_2-h_2\phi_1) \quad \text{See (21)} \quad \text{(Form III)} \\ &= r\Delta^2 h'\end{aligned} \tag{j}$$

which is seen to be invariant. It may also be written

$$\frac{\sigma'}{\Delta^3}=\beta h', \tag{j'}$$

as an absolute invariant.

The Jacobian of the cubic K and the quadratic θ may be written:

$$J_{K\theta}=-r^3\begin{vmatrix} Sx'x_1 & Sx'x_2 \\ Sa''x_1-Sx''a_1 & Sa''x_2-Sx''a_2\end{vmatrix}=\Delta r^3\left\{Sa''\frac{dx}{dn}-Sx''\frac{da}{dn}\right\}$$

(Forms I and III)

$$=\quad \Delta r^3\left\{\frac{d}{dn}\left(\frac{1}{\rho}\right)+2\frac{W\Delta\beta'}{r}+D\left(h-\frac{1}{\rho}\right)-D\left(h-\frac{1}{\rho}\right)\right\}$$

See (57) and (64)

$$=\quad \Delta r^3\left\{\frac{d}{dn}\left(\frac{1}{\rho}\right)+2\frac{W\beta'}{\beta}\right\}. \tag{k}$$

From this we have the absolute invariant form,

$$\frac{J_{K\theta}}{\Delta^4}=\beta^3\frac{d}{dn}\left(\frac{1}{\rho}\right)+2W\beta'\beta^2. \tag{k'}$$

As a special case of the quadratic

$$-Sx'x'=Eu'^2+2Fu'v'+Gv'^2$$

and the cubic $\qquad K=Pu'^3+3Qu'^2v'+3Ru'v'^2+Sv'^3=Sa''x'-Sx''a',$

we may write $\qquad J_{23}=\dfrac{\Delta}{\beta}\left\{\beta\dfrac{d}{dn}\left(\dfrac{1}{\rho}\right)+2W\beta'\right\}.$

The Jacobian of P and χ may be written

$$J_{P\chi}=\quad 6r^3\begin{vmatrix} -Sa_1x' & Sa_2x' \\ (Sa''x_1-Sx''a)_1- & (Sa''x_2-Sx''a_2)\end{vmatrix}$$

$$=\quad 6r^3\begin{vmatrix} Sa'x_1 & Sa'x_2 \\ Sa''x_1 & Sa''x_2\end{vmatrix}-6r^3\begin{vmatrix} Sa_1x' & Sa_2x' \\ Sx''a_1 & Sx''a_2\end{vmatrix}, \quad \text{since } Sa_1x'=Sa'x_1$$

$$=-6r^3\{SVx_1x_2Va'a''-SVa_1a_2Vx'x''\}=-6\Delta r^3\{Saa'a''-kSax'x''\} \quad \text{(Form I)}$$

$$=+6\Delta r^3\left\{\frac{1}{\rho}\frac{d}{dn}\left(\frac{1}{\rho}\right)+Dk+\frac{2W\beta'}{\rho\beta}-WP-Dk\right\} \quad \text{See (66) and (16)}$$

$$=\quad 6\Delta^3\left\{\frac{1}{\rho}\frac{d}{dn}\left(\frac{1}{\rho}\right)+\frac{2W\beta'}{\rho\beta}-WP\right\}. \tag{l}$$

$$\therefore\ \frac{J_{P\chi}}{\Delta^4}=\quad 6\beta^3\left\{\frac{1}{\rho}\frac{d}{dn}\left(\frac{1}{\rho}\right)+\frac{2W\beta'}{\rho\beta}-WP\right\}. \tag{l'}$$

The discriminant of the quadratic d may be given

$$\begin{aligned}
\Delta^2(AC-B^2) &= (\phi_1\Delta_1+Saw_1x_1)(\phi_2\Delta_2+Saw_2x_2)-(\phi_1\Delta_2+Saw_2x_1)(\phi_2\Delta_1+Saw_1x_2)\\
&=\Delta_1 Saw_2(\phi_1x_2-\phi_2x_1)+\Delta_2 Saw_1(\phi_2x_1-\phi_1x_2)+Saw_1x_1Saw_2x_2\\
&\quad -Saw_2x_1Saw_1x_2=Saw(\Delta_1w_2-\Delta_2w_1)+Saw_1\{x_1Saw_2x_2-x_2Saw_2x_1\}\\
&=Saw(\Delta_1w_2-\Delta_2w_1)+Saw_1\{w_2Sax_1x_2+x_2Saw_2x_1-x_2Saw_2x_1\}\\
&=Saw(\Delta_1w_2-\Delta_2w_1)-\Delta Saw_1w_2 \quad \text{(Forms II and III)}\\
&=-r^2\Delta'^2+\Delta r^2D\frac{d\Delta}{du}+r\Delta\Delta'r'-\Delta^3r'\beta'-\Delta^2\frac{dr}{du}rD\\
&\qquad\qquad\qquad\qquad\text{See (70) and (56)}\\
&=-\Delta^2rD\left(\frac{dr}{du}-\beta\frac{d\Delta}{du}\right)-r\Delta\beta\Delta'^2+\Delta r\Delta'(\Delta\beta'+\Delta'\beta)-\Delta^3r'\beta'\\
&=-\Delta^2rD\left(\frac{dr}{du}-\beta\frac{d\Delta}{du}\right)+\Delta^3\beta'(\beta\Delta'-r')\\
&=-\Delta^2rD\left(\frac{dr}{dn}-\beta\frac{d\Delta}{dn}\right)-\Delta^4\beta'^2=-\Delta^3rD\frac{d\beta}{dn}-\Delta^4\beta'^2. \qquad \text{(m)}
\end{aligned}$$

$$\therefore\ \frac{1}{\Delta^2}(AC-B^2)=-\beta'^2-\beta D\frac{d\beta}{dn}. \qquad \text{(m}')$$

The intermediate invariant of d and χ is given by

$$\begin{aligned}
NA-2MB+LC &= -\frac{1}{\Delta}[\{Sa_2x_2(\Delta_1\phi_1+Saw_1x_1)-Sa_2x_1(\Delta_1\phi_2+Saw_1x_2)\}\\
&\quad -\{Sa_1x_2(\Delta_2\phi_1+Saw_2x_1)-Sa_1x_1(\Delta_2\phi_2+Saw_2x_2)\}]\\
&=-\frac{1}{\Delta}[\Delta_1Sa_2(\phi_1x_2-\phi_2x_1)-\Delta_2Sa_1(\phi_1x_2-\phi_2x_1)\\
&\quad +Sa_2x_2Saw_1x_1-Sa_2x_1Saw_1x_2-Sa_1x_2Saw_2x_1+Sa_1x_1Saw_2x_2]\\
&=-\frac{1}{\Delta}\{(\Delta_2Swa_1-\Delta_1Swa_2)+\Delta\{Sa_2w_1-Sa_1w_2\}\} \quad \text{(Form III)}\\
&= \beta\Delta\left(W\Delta'-\frac{1}{\rho}\frac{d\Delta}{dn}\right)-\Delta rD\left(h-\frac{1}{\rho}\right)+\frac{\Delta}{\rho}\frac{dr}{dn}\\
&\qquad -2W\Delta^2\beta'-\Delta'rW \quad \text{See (68) and (72)}\\
&= \frac{\Delta^2}{\rho}\frac{d\beta}{dn}-\Delta^2\beta D\left(h-\frac{1}{\rho}\right)-2W\Delta^2\beta'. \qquad \text{(n)}
\end{aligned}$$

$$\therefore\ \frac{1}{\Delta^2}\left\{NA-2MB+LC\right\}=\frac{1}{\rho}\frac{d\beta}{dn}-\beta D\left(h-\frac{1}{\rho}\right)-2W\beta'. \qquad \text{(n}')$$

It is well to notice that the above gives the form

$$NA-2MB+LC=\Delta\left\{Sa_1\left(\frac{w}{\Delta}\right)_2-Sa_2\left(\frac{w}{\Delta}\right)_1\right\} \quad \text{(Form III).}$$

We have

$$\begin{vmatrix} E & F & G \\ L & M & N \\ A & B & C \end{vmatrix} = C(EM-FL)+B(LG-FM)+B(FM-EN)+A(FN-MG).$$

Now we find

$$\begin{aligned}
EM-FL &= -[Sx_1x_1Sa_1x_2 - Sx_1x_2Sa_1x_1] = -SVx_1x_2Va_1x_1 = -\Delta Saa_1x_1, \\
LG-FM &= -[Sx_2x_2Sa_1x_1 - Sx_1x_2Sa_1x_2] = SVx_1x_2Va_1x_2 = \Delta Saa_1x_2, \\
FM-EN &= -[Sx_1x_2Sa_2x_1 - Sx_1x_1Sa_2x_2] = SVx_1x_2Va_2x_1 = \Delta Saa_2x_1, \\
FN-MG &= -[Sx_1x_2Sa_2x_2 - Sa_2x_1Sx_2x_2] = -SVx_1x_2Va_2x_2 = -\Delta Saa_2x_2.
\end{aligned}$$

$$\begin{aligned}
\therefore \begin{vmatrix} E & F & G \\ L & M & N \\ A & B & C \end{vmatrix} &= Saa_1\{x_1(\phi_2\Delta_2+Saw_2x_2)-x_2(\phi_1\Delta_2+Saw_2x_1)\} \\
&\quad -Saa_2\{x_1(\phi_2\Delta_1+Saw_1x_2)-x_2(\phi_1\Delta_1+Saw_1x_1)\} \\
&= \Delta_2Saa_1w-\Delta_1Saa_2w+Saa_1\{w_2Sax_1x_2+x_2Saw_2x_1-x_2Saw_2x_1\} \\
&\quad -Saa_2\{w_1Sax_1x_2+x_2Saw_1x_1-x_2Saw_2x_1\} \\
&= Saw(\Delta_1a_2-\Delta_2a_1)+\Delta(Saa_2w_1-Saa_1w_2) \quad \text{(Form III)} \\
&= r\Delta'\Delta h - r\Delta W\frac{d\Delta}{dn} - r\frac{\Delta}{\rho}\Delta' + \Delta^2W\left(rD+\frac{dr}{dn}\right) \\
&\qquad + \frac{\Delta^2}{\rho}\beta\Delta' + \frac{2\Delta^3\beta'}{\rho} - r'\Delta^2h \quad \text{See (69) and (71)} \\
&= \Delta^2h(\beta\Delta'-r') - \Delta^2W\left(\beta\frac{d\Delta}{dn}-\frac{dr}{dn}\right)+\Delta^2WrD+\frac{2\Delta^3\beta'}{\rho} \\
&= -\Delta^3h\beta'+\Delta^3W\frac{d\beta}{dn}+\Delta^3\beta WD+\frac{2\Delta^3\beta'}{\rho} \\
&= \Delta^3\left[\left(\frac{2}{\rho}-h\right)\beta'+W\left\{\frac{d\beta}{dn}+\beta D\right\}\right]. \qquad \text{(o)}
\end{aligned}$$

$$\therefore \frac{1}{\Delta^3}\begin{vmatrix} E & F & G \\ L & M & N \\ A & B & C \end{vmatrix} = \left(\frac{2}{\rho}-h\right)\beta'+W\left(\frac{d\beta}{dn}+\beta D\right). \qquad \text{(o')}$$

Many other examples could be given, but enough has been shown to illustrate the use of vector methods in discussing certain forms found in differential geometry. The reader will see how much more direct and simple these expressions and reductions are in vector forms than in the use of the more tedious expressions in Cartesian coordinates.

UNIVERSITY OF TEXAS.